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Quadrature algorithms to the luminosity distance with a time-dependent dark energy model

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Abstract. In our previous work [1], we have proposed two methods for computing the luminosity distance d_L^Λ in ΛCDM model. In this paper, two effective quadrature algorithms, known as Romberg Integration and composite Gaussian Quadrature, are presented to calculate the luminosity distance d_L^{CPL} in the Chevallier-Polarski-Linder parametrization(CPL) model. By comparing both the efficiency and accuracy of the two algorithms, we find that the second is more promising. Moreover, we develop another strategy adapted for approximating d_L^Λ in flat ΛCDM universe. To some extent, our methods can make contributions to the recent numerical stimulation for the investigation of dark energy cosmology.

Contents

1 Introduction

The computation and numerical evaluation of distances is frequently encountered in the research of cosmological phenomena. In practice, it is common to compute the various cosmological distances as a function of the redshift z under certain cosmological models. Current cosmological observations indicate that the expansion of universe is accelerating, has a prominent dark energy content ΛCDM $\simeq 0.7$ and is spatially flat [2]. Further, a Lambda cold dark matter (ΛCDM) model fits the data well, and is frequently used as a fiducial or background model. In the ΛCDM model, various cosmological distances can be expressed in terms of the elliptic integrals [3, 4].

Some works have focused on the luminosity distance d_L in the ΛCDM model and derived numerical approximations for the efficient and accurate evaluation of $d_L(z)$ given the cosmological parameters Ω_m and Ω_Λ [5–7]. The computation of d_L is useful in the analysis of distance-redshift relations of type Ia supernovae, and the approximation for d_L can also be directly used in the evaluation of other distances, for instance the angular diameter distance d_A or the comoving distance r [4].

Recent years, Chevallier, Polarski [8] and Linder [10] proposed a simple parametrization of the dark energy equation of state (known as CPL):

$$w(z) = w_0 + w_a \frac{z}{1+z}, \tag{1.1}$$

which is involved in the luminosity distance. The best fit values of w_0 and w_a are -1.58 and 3.30[11], respectively. CPL parametrization is widely applied into both observational and theoretical analysis, and it has the talent to test the dynamics of many dark energy models. More discussions about CPL model can be seen [9–13]. In this paper, we just concentrate on the numerical analysis of the complicated integral contained in the luminosity distance of CPL parametrization model. Note that analogous integral may be encountered in many cases, e.g. the dynamical age of the universe or the angular diameter distance.

For the rest of this paper we will pay our main attention to perform the numerical quadrature algorithms. Section 2 is a brief review of the luminosity distance in the CPL parametrization model. In section 3, we present two different quadrature algorithms and

compare their efficiency and accuracy based on the personal computer. Another approximate recipe of the luminosity distance in ΛCDM is developed in section 4. Finally, we discuss some improvements of the algorithms and possible extensions to other cosmological models briefly.

2 Luminosity Distance in CPL Model

In order to study the different dark energy models, the widely used method is assume an *ad hoc* equation of state $w(z) = p_X/\rho_X$ for dark energy and parametrize $w(z)$ [14]. CPL parametrization model was first proposed by M. Chevallier, D. Polarski [8] and E. V. Linder [10], and the parameterized $w(z)$ can be written as equation 1.1. Thus the dark energy density ρ_X is given by

$$\rho_X(z) = \rho_X^0 f(z) \tag{2.1}$$

with

$$f(z) = (1+z)^{3(1+w_0+w_a)} \exp(-\frac{3w_a z}{1+z}). \tag{2.2}$$

For a spatially flat universe ($k = 0$), the Friedmann equation can be expressed as

$$
\begin{aligned}
H^2(z) &= \frac{8\pi G}{3}(\rho_M + \rho_X) = H_0^2 E^2(z) \\
&= H_0^2[\Omega_{\mathrm{m}}(1+z)^3 + \Omega_\Lambda(z)],
\end{aligned}
\tag{2.3}
$$

where $H(z)$ is the Hubble parameter, Ω_{m} is the dark matter parameter, $\Omega_\Lambda(z)$ represents the time-dependent dark energy parameter, $E(z)$ is the expansion rate of the universe.

With the continuous equation and Friedmann equation 2.3, $\Omega_\Lambda(z)$ can be deduced as

$$\Omega_\Lambda(z) = \Omega_\Lambda f(z), \tag{2.4}$$

where Ω_Λ is the dark energy parameter at present time and $f(z)$ is defined as equation 2.2. Hence, the luminosity distance in the CPL model can be written in the form

$$d_L^{CPL} = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{E(z')}. \tag{2.5}$$

The expression of the luminosity distance is complicate and there is no critically analytical solution for general parameter choice (w_0, w_a). Meanwhile, we can take the luminosity distance in ΛCDM universe as a special case with $(w_0, w_a) = (-1.0, 0.0)$. From this point of view, the d_L^{CPL} will degenerate to

$$d_L^\Lambda = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{\mathrm{m}}(1+z')^3 + \Omega_\Lambda}}. \tag{2.6}$$

3 Quadrature Algorithms

Equation 2.5 is just a one-dimensional integral, but there are four variables in the integrand, i.e. Ω_{m}, z, w_0, w_a. Because of the different variations of the three variables $(\Omega_{\mathrm{m}}, w_0, w_a)$, approximating the integral directly seems to be impossible if we want to obtain desirable

accuracy. On the other hand, if we approximate the integrand with multivariables interpolation technique [15] and then integrate the approximate polynomial, the expression will remain complicate and is hardly to implement in practice.

Instead of developing an analytical approximation, an effective quadrature algorithm may be more helpful. We present two conventional numerical integration methods, known as Romberg Integration and Gaussian Quadrature [16], in the following subsections and compare their performances to see which one is more suitable for calculating the luminosity distance.

For simplifying the following description, we just consider the integral in equation 2.5, defined as :

$$fE = \int_0^z \frac{dz'}{E(z')} \tag{3.1}$$

with

$$E^2(z) = \Omega_{\rm m}(1+z)^3 + \Omega_\Lambda(1+z)^{3(1+w_0+w_a)} \exp(-\frac{3w_a z}{1+z}),$$

where $\Omega_{\rm m} + \Omega_\Lambda = 1$.

3.1 Romberg Integration

Romberg Integration gives preliminary approximations with the Composite Trapezoidal rule and then applies the Richardson extrapolation process to improve the accuracy. For each integer $k = 2, 3, 4, ..., n$ and $j = 2, 3, ..., k$, an $O(h_k^{2j})$ approximation formula can be written as

$$\int_a^b f(x)dx = R_{k,j} + O(h_k^{2j}), \tag{3.2}$$

where $h_k \equiv (b-a)/2^{k-1}$ and the iterative formula $R_{k,j}$ is

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}. \tag{3.3}$$

In order to obtain the complete iterative process, we should considerate the case when $j = 1$. In general, the $R_{k,1}$ is provided by using the trapezoidal approximation, then we have

$$R_{1,1} = \frac{h_1}{2}[f(a) + f(b)], \tag{3.4}$$

and

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)\right], \tag{3.5}$$

for $k = 2, 3, ..., n$.

The main results generated from the above formulas are listed in Table 1. The Romberg Integration has an additional desirable feature that it allows an entire new row in the table to be calculated by performing one additional application of the Composite Trapezoidal rule. Then it uses an averaging of the previously calculated values to obtain the succeeding entries in the row. The method used to construct a table of this type calculates the entries row by

Table 1. Approximation results for Romberg Integration.

$R_{1,1}$					
$R_{2,1}$	$R_{2,2}$				
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$

row, that is, in the order $R_{1,1}$, $R_{2,1}$, $R_{2,2}$, etc. R. L. Burden describes a detailed algorithm in [17].

We can preset an integer n to determine the number of rows. In many cases, however, it is confused to ensure whether the output is satisfactory or too many entries are unnecessary to generate. For using the iterative technique sufficiently and saving the running time, we can set an error tolerance for the approximation and generate n, within some upper bound, until some consecutive entries agree to within the tolerance. In this paper, we choose $|R_{n,n} - R_{n,n-1}| < 10^{-6}$ to generate the approximations.

The Romberg Integration method can be used in conjunction with other numerical quadrature formulae to obtain successive improved values. Further, the method is applicable to a very large class of functions.

3.2 Composite Gaussian Quadrature

Gaussian Quadrature chooses the points for evaluation in an optimal, rather than equally spaced. The nodes $r_1, r_2, ..., r_n$ in the interval $[a, b]$ and coefficients $c_1, c_2, ..., c_n$, are chosen to minimize the expectancy obtained in the approximation

$$\int_a^b f(x)dx \approx \sum_{i=1}^n c_i f(r_i). \tag{3.6}$$

With the Legendre polynomials, we can determine the nodes and coefficients easily. The nodes r_i are the zeros of the nth Legendre polynomial $P_n(x)$. Table 2 lists the nodes and coefficients for n=7 and 8. More detailed calculations and values of nodes and coefficients can be found in [16, 18]. Such quadrature formulae is called Gauss-Legendre formulae.

The error term of higher-order derivative in quadrature formulae of higher degree is difficult to evaluated or even boundless, the quadrature formulae 3.6 is not recommended to obtain desirable accuracy, although the Gaussian Quadrature is stable. Instead, we can divide the interval $[a, b]$ into some subintervals $[x_i, x_{i+1}]$, and apply the low-order Gaussian Quadrature to each subinterval. Then, summing over all the values as the final approximation, we have

$$\int_a^b f(x)dx = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x)dx, \tag{3.7}$$

where $a = x_0 < x_1 < ... < x_m = b$ and the subscript m donates the number of the subintervals.

Table 2. Nodes and weight coefficients for Gauss-Legendre integration.

n	nodes $r_{n,i}$	Coefficients $c_{n,i}$
7	$\pm$ 0.9491079123	0.1294849662
	$\pm$ 0.7415311856	0.2797053915
	$\pm$ 0.4058451514	0.3818300505
	0	0.4179591837
8	$\pm$ 0.9602898565	0.1012285363
	$\pm$ 0.7966664774	0.2223810345
	$\pm$ 0.5255324099	0.3137066459
	$\pm$ 0.1834346425	0.3626837834

Let the subintervals be of equal size, the composite Gaussian Quadrature can be written as:

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{i=1}^{n} c_i \left[\sum_{j=0}^{m-1} f(a + \frac{2j+1}{2}h + \frac{h}{2}r_i)\right], \qquad (3.8)$$

where $h = (b-a)/m$ donates the size of the subinterval.

The Gaussian Quadrature formulae can be applied only when $f(x)$ is explicitly known, so that $f(x)$ can be evaluated for any desired value of x. Naturally, orthogonal polynomial other than the Legendre polynomials also can be used, such as the Gauss - Chebyshev, Gauss - Jacobi and Gauss - Hermite formulae.

3.3 Performance of the Two Algorithms

In the section, we perform the efficiency and accuracy of the two quadrature algorithms with Fortran program. We create a sample containing about 10^5 redshift data which ranges from 0 to 1100 to present a quantitative comparison.

The Romberg Integration is based on equation 3.3, and we choose the error tolerance $|R_{n,n} - R_{n,n-1}| < 10^{-6}$ to generate the approximation. With the fast convergence rate of the iterative, there is no obvious distinctness, including the execution time, if we set the error tolerance $|R_{n,n} - R_{n,n-1}| < 10^{-4}$ instead. Different parameters (m, w_0, w_a) are chosen to evaluate the composite Gaussian Quadrature formulae 3.8, but we fix the number of nodes $n = 8$.

Fig 1 shows the relative error of different parameter choices for Gaussian Quadrature which illustrates that more subinterval division can improve the accuracy of the algorithm and the relative error also depends on the choice of w_0 and w_a. However, the error is insensitive to the m for redshift $z < 50$. Actually, the general Gaussian Quadrature based on equation 3.6, i.e. $m = 1$, is precise enough to evaluate the integral values in this case. Therefore, in order to obtain desirable accuracy and efficiency, we can set a greater number n and suitable m to extend the composite Gaussian Quadrature to a wider redshift distribution.

The main results of the two algorithms are listed in Table 3. Note that the Gaussian Quadrature obviously takes less time than Romberg Integration if the same accuracy is required. However, extra interpretation about Table 3 should be emphasized. The execution time and efficiency just reflects the relative results of the codes with each other, and depends on the compiler as well as the different computing environment. More discussions are available in [1] and [6].
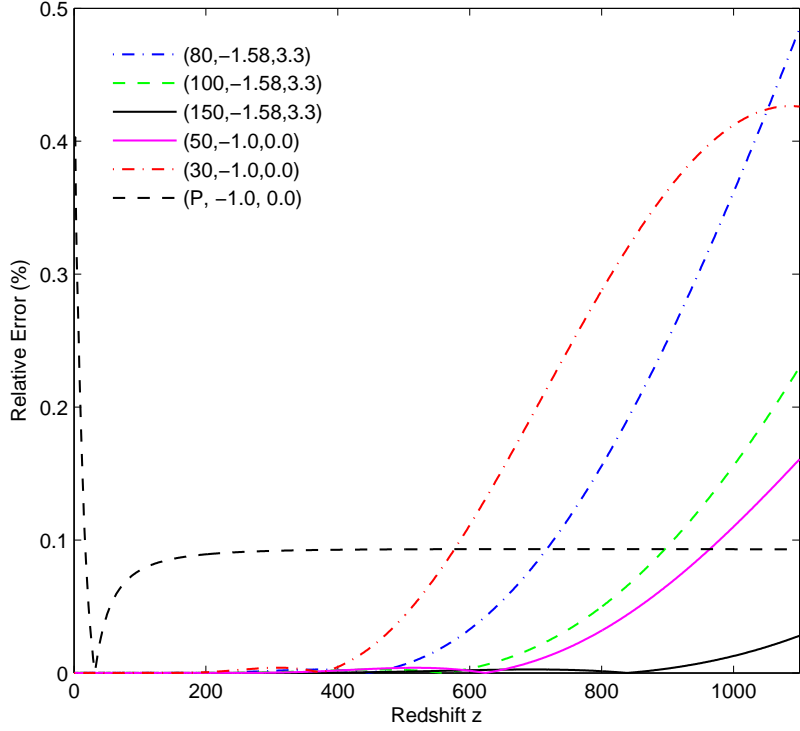
Figure 1. The absolute relative error as a function of z for different parameter choices (m, w_0, w_a). The error is no more than 0.5% when $m > 80$ for the best fit parameters $w_0 = -1.58$, $w_a = 3.3$, $\Omega_m = 0.3$. The black dashed line, i.e. (P, -1.0, 0.0), denotes the accuracy of the polynomial approximation (see section 4).

Table 3. The main results of the Romberg Integration, Gaussian Quadrature and polynomial approximation. The last two columns show that the iterative of Romberg Integration spends more time, although its accuracy is well under control.

type	parameters	time(s)	maximum error(%)
Romberg Integration	(-1.58, 3.3)	210.960	0.04
Gaussian Quadrature	(80, -1.58, 3.3)	7.457	0.48
	(100, -1.58, 3.3)	9.266	0.23
	(150, -1.58, 3.3)	13.822	0.028
	(200, -1.58, 3.3)	18.564	0.003
	(30, -1.00, 0.0)	1.888	0.43
	(50, -1.00, 0.0)	3.151	0.16
Polynomial approximation	(P, -1.0, 0.0)	0.016	0.41

4 A Polynomial Approximation to d_L^Λ

The general analytical expression of the luminosity distance in the spatially flat ΛCDM model is given by equation 2.6. Because it is frequently used in practice, many papers have focused on the numerical analysis of the integral equation. In this section, another polynomial

approximation is described for the similar considerations. Following the notation introduced in our previous work [1], substituting $s = \sqrt[3]{(1 - \Omega_m)/\Omega_m}$ and $u = 1/z'$ into equation 2.6 yields

$$\frac{d_L^\Lambda}{c/H_0} = \frac{1+z}{\sqrt{s\Omega_m}} \left[T(s) - T(\frac{s}{1+z}) \right],$$

where

$$T(\tau) = \int_0^\tau \frac{du}{\sqrt{u^4 + u}}. \tag{4.1}$$

However, we note that the behavior of $T(\tau)$ has some deficiencies. First, the derivative of $T(\tau)$ becomes singular as $\tau \to 0^+$. Second, the domain of $T(\tau)$ extends to infinity. Either one is detrimental to the approximation using polynomials. Fortunately, such shortages can be eliminated by proper change of variables. Based on the consideration above, we introduce a mathematical transformation $x = 1/(\tau + 1)$ further to constraint the variation of x within $0 \le x \le 1$, which leads to

$$\frac{d_L^\Lambda}{c/H_0} = \frac{1+z}{\sqrt{s\Omega_m}} \left[f(\frac{1}{s/(1+z)+1}) - f(\frac{1}{s+1}) \right],$$

where

$$f(x) = \int_0^x \frac{du}{\sqrt{(1-u) \cdot (3u^2 - 3u + 1)}}. \tag{4.2}$$

The main purpose of the section is to obtain the approximate expression of the equation.

Since the integrand intends to infinity as $u \to 1^-$, we utilize a roundabout strategy to achieve the suitable polynomial. By analyzing the integrand, we find that it can be factorized as above (equation 4.2), which inspires us to write it as:

$$\hat{f}(x) = \int_0^x \frac{a_i u^i du}{\sqrt{1-u}}, \tag{4.3}$$

where $a_i u^i$ is a polynomial defined as: $a_i u^i = \sum_{i=0}^5 a_i u^i$.

One prominent aspect of the equation 4.3 is that it can be integrated analytically. For quarrying out the six best-fitting free parameters a_i, we must impose some constraints upon the equation 4.3. With the modish range of Ω_m in mind, we just minimize the relative error between function $\hat{f}$ and f with $0.1 < \Omega_m < 1$. The restrictions what we adopt are listed as following:

$$\hat{f}(\frac{1}{3}) = f(\frac{1}{3}), \quad \hat{f}(1) = f(1),$$
$$\hat{f}'(\frac{1}{3}) = f'(\frac{1}{3}), \quad \hat{f}'(1) = f'(1), \tag{4.4}$$

where $1/3$ is approximately equal to the minimum value of x within the considered parameter space, i.e., $x \in [1/3, 1]$.

Utilizing equation 4.4, we can obtain a linear equations which contains just two free parameters. For instance, we can choice a_4 and a_5 as the variables to be determined. In order to derive the total parameters, we define the relative error as:

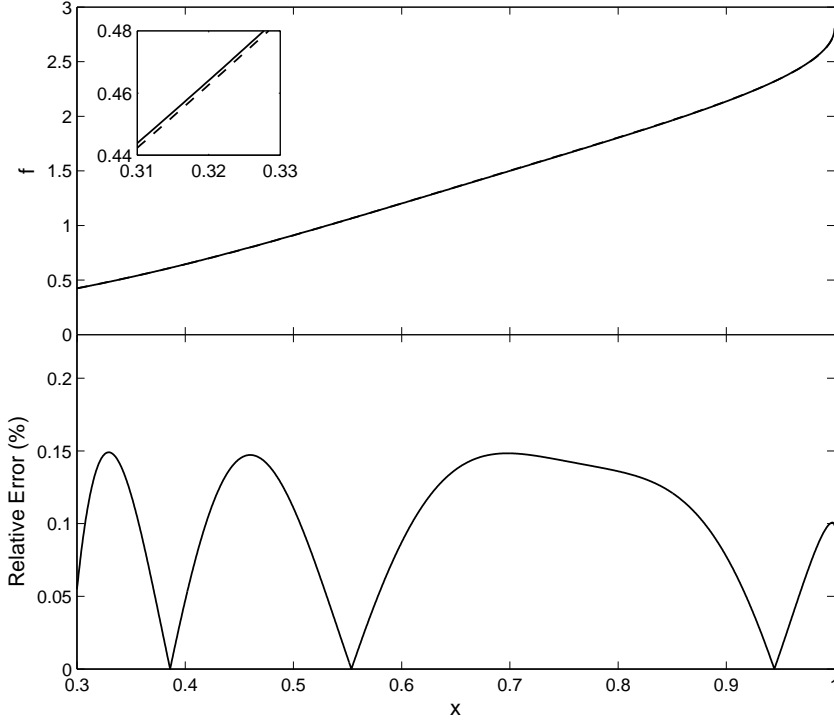$$e = |\frac{\hat{d}_L^\Lambda}{d_L^\Lambda} - 1|. \tag{4.5}$$

Figure 2. Approximate and analytical function f (*upper panel*) and the absolute percentage relative error (*lower panel*). The solid and dashed lines represent the $\hat{f}$ and f with $0.1 \leq \Omega_{\mathrm{m}} \leq 1$ in the upper panel, respectively.

Just as [5] has pointed out, the error tends to be dominated by $z \to 0$, which yields

$$e(x)_{max} = |\frac{d\hat{f}}{df} - 1|_{z \to 0}. \tag{4.6}$$

The square of $e(x)$, namely $e^2(x)$, as a continuous function, is more convenient for us to acquire the relation between the rest two parameters a_4 and a_5. From the mathematical theorem we know that the first derivative of $e^2(x)$, which contains three variables x, a_4 and a_5, must be equal to zero strictly if it reaches the local maximum. By solving the three nonlinear equations, the most appropriate relation between a_4 and a_5 can be written as following:

$$a_4 = 12.15722 - 2.92471 \cdot a_5, \tag{4.7}$$

where a_5 is still unknown to us. Because we are more interested in the minimum of the $e(x)_{max}$, substituting equation 4.7 into equation 4.6 and regulating the valve of a_5 make us find the most desirable outcome of $e(x)_{max}$ to be about 0.96% when $a_5 = -44.63290$.

However, we have emphasized that the implicit requirement of the coincidence of function values at end points could be unnecessarily strong [1]. If we control the relative error to the minimum when Ω_{m} is taken as the best observational value 0.3, the final approximation
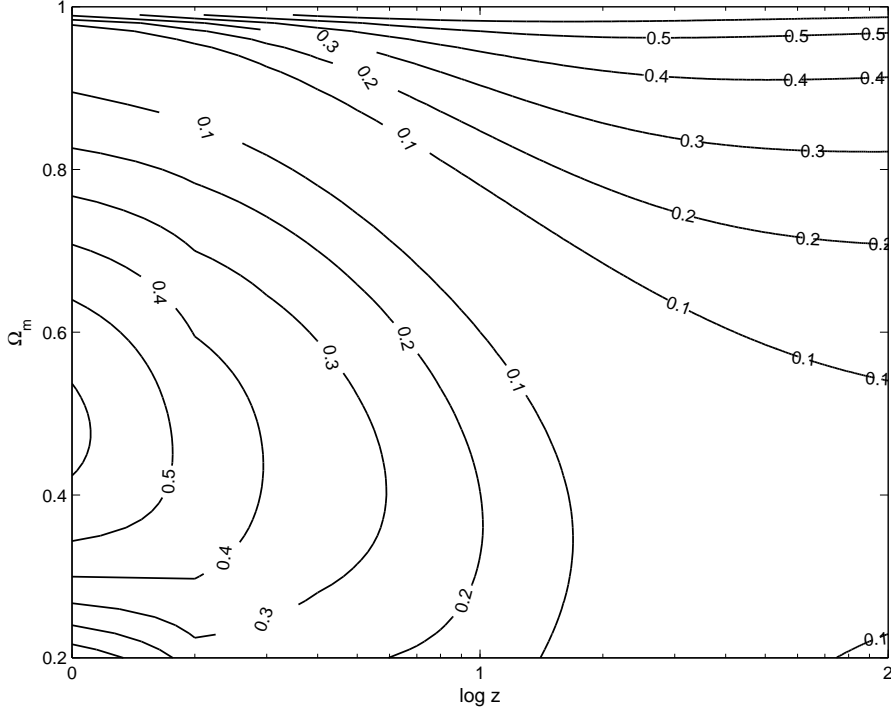
Figure 3. The distribution of the global relative error in d_L with different z and Ω_{m}. The maximum error, dominated by the small redshift ($z < 0.1$), is less than 1%.

we construct can be expressed as:

$$\hat{f}(x) = \sqrt{1-x} \cdot (8.11507x^5 - 22.69338x^4 + 21.09474x^3$$
$$-6.03039x^2 + 0.32109x - 2.80713) + 2.80713, \tag{4.8}$$

where

$$x = \frac{1}{s/(1+z)+1}, \quad s^3 = \frac{1-\Omega_{\mathrm{m}}}{\Omega_{\mathrm{m}}}.$$

Fig. 2 shows the results of the fit for the function f (see equation 4.2) and its residual, which is not more than 0.15%.

With the definition of the relative error in equation 4.5, the distribution of the global relative error based on the polynomial approximation (equation 4.8) with various redshift and Ω_{m} is plotted in Fig. 3. As we can see from the figure, the maximum error, which is less than 1%, tends to be contributed by the small redshift z. Similarly, we compute the running time and accuracy of the approximate luminosity distance based on the same criterions presented in section 3.3. From Table 3 we conclude that the polynomial spends less time than the other two methods, though its error can't be improved further. However, Fig. 1 shows that the accuracy of the polynomial at low redshift is inferior to the composite Gaussian Quadrature. And as the increase of redshift z, the relative error tends to be around 0.09%.

Because approximating the integrand directly can decrease the error of the target-integral, one promising extension of our method is to the linear growth factor $\delta(a)$ [19, 20] contained only Ω_{m} and Ω_{Λ}. The general form of $\delta(a)$ in ΛCDM universe can be written as

$$\delta(a) = \frac{5\Omega_{\mathrm{m}}}{2} \frac{H(a)}{H_0} \int_0^a \frac{da'}{[a'H(a')/H_0]^3}, \tag{4.9}$$

where a and $H(a)$ represent the scale factor and Hubble parameter, respectively. Kasai [7] has developed an effective recipe for evaluating it recently. However, a more compact form may be helpful to research the matter density perturbation with numerical simulation efficiently, and as a candidate our idea behind the method may be useful.

5 Conclusions

Two different quadrature algorithms are presented to evaluate the integral involved in luminosity distance in CPL parametrization model. Through the comparison of the efficiency and accuracy, the composite Gaussian Quadrature is more promising to apply into the evaluation of such complex integral. Because of the generalization of the algorithms, we can also extend them to other parametric models of dark energy, for instance the two-index parameterizations by Huterer et al.[22], and the four-index parameterizations by Hannestad et al.[23].

Additionally, because general Gaussian Quadrature is precise enough for refshift $z < 50$, it is unnecessary to divide the interval $[a, b]$ to be of equal size. The adaptive integration combined with the composite Gaussian Quadrature may be more helpful.

Acknowledgments

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